Econ 802
Final Exam Answers

Greg Dow December 2019

1. (a) It is true that IRS implies a decreasing average cost curve in the long run. It is also true for a price-taking firm that if AC ever falls below the output price, the firm can always increase profit by increasing output. But this does not mean that IRS never happens in the real world. It just means that we can't use a price-taking model in markets of this kind. Instead we could use a monopoly or oligopoly model, where a firm has finite scale despite the existence of IRS.

(b) This statement confuses two separate ideas. We often assume output is a concave function of the input quantities, because this helps to guarantee the existence of a profit maximizing production plan (for instance, in a Cobb-Douglas production function with decreasing or constant returns). But even with a concave production function, it is still true that the profit function is convex in prices. The reason is that when prices change, the firm can always respond passively, not changing its behavior. This gives a linear change in profit. But if the firm responds optimally to the price change, it can usually do better, and this gives the convexity.

1(c) This is not correct in general. Suppose we had a U-shaped SAC curve, and the LAC passed through the minimum of SAC, and LAC is falling. Then at output levels just to the left of the min, we would have LAC > SAC. But this is impossible because cost can't be higher in the long run. In the same way, if LAC is rising, at output levels just to the right of the min, again we have LAC > SAC. The only way in which LAC can intersect the min of an SAC curve without a contradiction is if LAC is horizontal, or LAC is also at a min.

2 (a) The Lagrangian is $L = w_1 x_1 + w_2 x_2 - \lambda\left[f(x_1, x_2) - y\right]$

FOC:
$$\left.\begin{aligned} w_1 - \lambda f_1(x) &= 0 \\ w_2 - \lambda f_2(x) &= 0 \end{aligned}\right\} \Rightarrow \frac{w_1}{w_2} = \frac{f_1(x)}{f_2(x)}$$
$$f(x) = y$$

$$\Rightarrow \frac{w_1}{w_2} = \frac{x_1}{x_2} \Rightarrow x_2 = \frac{x_1 w_2}{w_1}$$

Where $f_1(x) = \frac{1}{2}\left[x_1^2 + x_2^2\right]^{-1/2} x_1 (2)$
$f_2(x) = \frac{1}{2}\left[x_1^2 + x_2^2\right]^{-1/2} x_2 (2)$

Substitute into constraint $\Rightarrow \left[x_1^2 + x_2^2\right]^{1/2} = y$

$$\Rightarrow \left[x_1^2 + \frac{x_1^2 w_2^2}{w_1^2}\right]^{1/2} = y$$

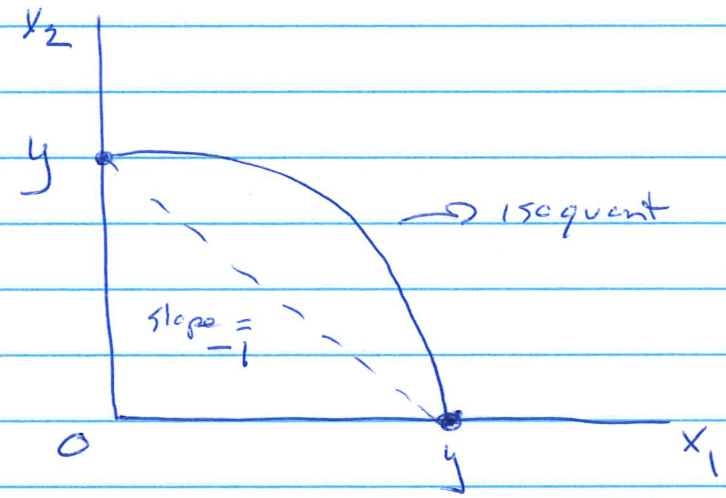
$$\Rightarrow x_1 \left[1 + \left(\frac{w_2}{w_1}\right)^2\right]^{1/2} = y$$

$$\Rightarrow x_1(w, y) = \frac{y}{\left[1 + \left(\frac{w_2}{w_1}\right)^2\right]^{1/2}} \qquad x_2(w, y) = \frac{y}{\left[1 + \left(\frac{w_1}{w_2}\right)^2\right]^{1/2}}$$

2 (a) continued.

The unusual thing about this solution is that when $w_1 \uparrow$ we get $x_1 \uparrow$ and when $w_2 \uparrow$ we get $x_2 \uparrow$. This implies upward sloping conditional input demand functions which contradicts our usual expectation that input demand curves will slope down.

(b)



It is clear from the production function that one way to get y is with $x_1 = y$ and $x_2 = 0$. Another way is $x_1 = 0$ and $x_2 = y$. So these boundary points are on the isoquant.

The question is what happens at interior points. We know the ~~marginal~~ ~~rate~~ technical rate of substitution is the absolute value of the slope of the isoquant, where

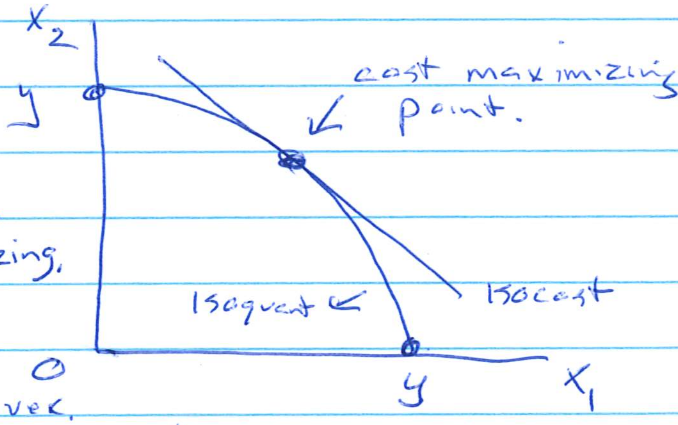
$$TRS = \frac{MP_1}{MP_2} = \frac{f_1(x)}{f_2(x)} = \frac{x_1}{x_2}.$$

As we approach the horizontal boundary $(y, 0)$, this $\to +\infty$. As we approach the vertical boundary $(0, y)$, this $\to 0$. So we get an isoquant like the one shown in the graph, where the slope is steeper as we move down and to the right.

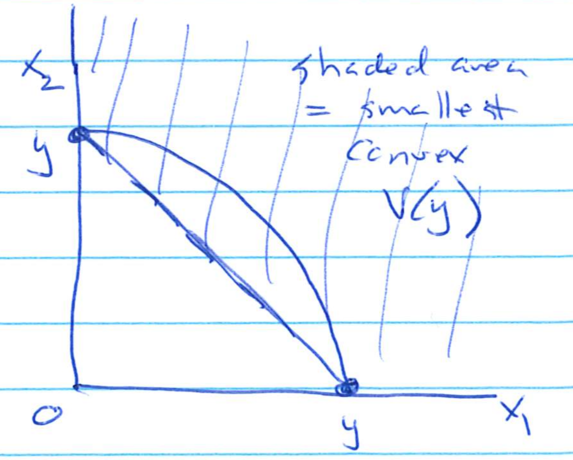
Now think about cost min. The slope of an isocost line is $-\frac{w_1}{w_2}$. If $w_2 > w_1$ this slope is flatter than -1 and the cost minimizing point is at the boundary $(y, 0)$. If $w_2 < w_1$, The slope of the isocost lines is steeper than -1, and the cost minimizing point is at the other boundary $(0, y)$. If $w_1 = w_2$ the firm is indifferent between $(y, 0)$ and $(0, y)$ because they are on the same isocost line (but the firm will not use any point on the dashed line between them). Thus

$x_1(w,y) = y$ and $x_2(w,y) = 0$ for $w_1 < w_2$
$x_1(w,y) = 0$ and $x_2(w,y) = y$ for $w_1 > w_2$
Indifferent between these points for $w_1 = w_2$

2(c) The solution in (a) was incorrect because although the FOC were satisfied, the necessary SOC was not. This is easy to see in a graph:



Clearly the production function is not quasi-concave, and we are at a cost maximizing point instead of cost minimizing. Alternatively, we could say the input requirement set is non-convex.



The smallest closed, convex, and monotonic input requirement set consistent with firm behavior as in part (b) is this: It must include $(y, 0)$ and $(0, y)$, plus all points on the line segment between them (due to convexity), plus all points to the northeast (due to monotonicity) including all boundary points along the axes (to make the set closed). You can check that for this $V(y)$, the firm behaves as in part (b).

3. (a) Slutsky equation:

$$\frac{\partial x_i(p,m)}{\partial p_i} = \underbrace{\frac{\partial h_i[p, v(p,m)]}{\partial p_i}}_{\text{substitution effect}} \underbrace{- \frac{\partial x_i(p,m)}{\partial m} x_i(p,m)}_{\text{income effect (including the minus sign)}}$$

A Giffen good has $\frac{\partial x_i}{\partial p_i} > 0$ so the demand curve slopes up.

Because the substitution effect always has $\frac{\partial h_i}{\partial p_i} \leq 0$, to get a Giffen good we need

$-\frac{\partial x_i}{\partial m} \cdot x_i > 0$. Due to $x_i \geq 0$, this can only happen when $\frac{\partial x_i}{\partial m} < 0$

So the good must be inferior (Marshallian demand drops when income rises) and the income effect must be so large that it outweighs the substitution effect.

(b) Choose any $p^* > 0$ and $m^* > 0$. Let x^* be optimal for (p^*, m^*) and write $u^* = u(x^*) = v(p^*, m^*)$.

Now consider the identity $\underbrace{h_i(p,u)}_{\text{Hicksian}} \equiv \underbrace{x_i[p, e(p,u)]}_{\text{Marshallian}}$

Given local non-satiation, this is true for all p from duality: the same consumption bundle solves the expenditure min problem for (p,u) and the utility max problem at the same prices when the consumer has the min income $e(p,u)$ needed to reach the utility level u.

Differentiate both sides to get

$$\frac{\partial h_i(p,u)}{\partial p_i} = \frac{\partial x_i[p, e(p,u)]}{\partial p_i} + \frac{\partial x_i[p, e(p,u)]}{\partial m} \frac{\partial e(p,u)}{\partial p_i}$$

From Shephard's Lemma, we can substitute

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

Now evaluate all derivatives at (p^*, m^*) and use $u^* = v(p^*, m^*)$ along with $h_i(p^*, u^*) = x_i(p^*, m^*)$. This gives

also $m^* = e(p^*, u^*)$

$$\frac{\partial h_i[p^*, v(p^*, m^*)]}{\partial p_i} = \frac{\partial x_i(p^*, m^*)}{\partial p_i} + \frac{\partial x_i(p^*, m^*)}{\partial m} x_i(p^*, m^*)$$

We can drop the stars because (p^*, m^*) was arbitrary. Rearranging the result gives the Slutsky equation in (a).

3(c) Matrix version of Slutsky equation:

$$\frac{\partial x(p,m)}{\partial p} = \frac{\partial h[p, v(p,m)]}{\partial p} - \frac{\partial x(p,m)}{\partial m} x(p,m)$$

$n \times n$ $\qquad$ $n \times n$ $\qquad$ $n \times 1$ (column vector) $\qquad$ $1 \times n$ (row vector)

$$\begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial p_1} & \cdots & \frac{\partial x_n}{\partial p_n} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_n} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial x_1}{\partial m} \\ \frac{\partial x_n}{\partial m} \end{bmatrix} \quad [x_1(p,m) \ldots x_n(p,m)]$$

Write this as

$$\frac{\partial h[p, v(p,m)]}{\partial p} = \underbrace{\frac{\partial x(p,m)}{\partial p} + \frac{\partial x(p,m)}{\partial m} x(p,m)}$$

empirical prediction: this is symmetric and negative semi-definite. ($\Rightarrow$ diagonal elements non-positive)

Reason: from Shephard's Lemma, $\frac{\partial h}{\partial p}$ is the Hessian matrix of the expenditure function $e(p,u)$. This Hessian must be symmetric. Furthermore we know expenditure functions are concave in prices, so the Hessian is also negative semi-definite. [one implication of this is that the diagonal elements cannot be positive]

4 (a) Demand for a typical consumer:

max $x(a-x)+y$ subject to $px+y=w$

$\Rightarrow$ max $x(a-x)+w-px$

Note: SOC sufficient condition always holds here.

FOC: $a-2x-p=0$ (Note: This is for the case $x>0$).

If the derivative of the objective function is ≤ 0 at $x=0$, then $x=0$ is optimal; this is true when $a \leq p$.

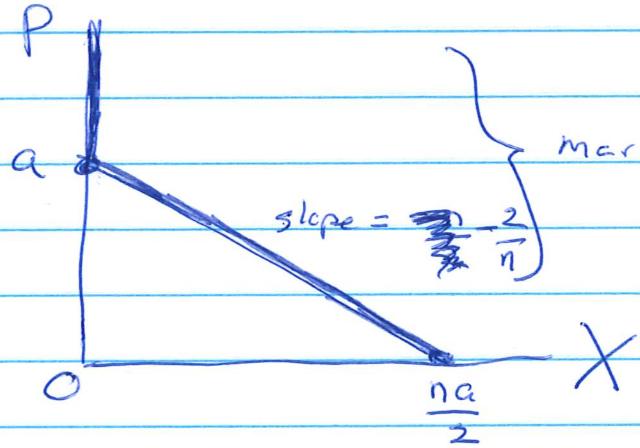
So the full demand function is $\begin{cases} x=0 & \text{for } p \geq a \\ x=\frac{a-p}{2} & \text{for } 0 \leq p \leq a \end{cases}$

For market demand, multiply by n consumers to get

$X=0$ for $p \geq a$

$X=n\frac{(a-p)}{2}$ for $0 \leq p \leq a$

There is a finite upper bound on X because $p \geq 0$ implies $X \leq \frac{na}{2}$ (this comes from the fact that utility is quadratic in x so marginal utility hits zero at $x=\frac{a}{2}$.)



(b) Supply for a typical firm: max $px-c(x)$

FOC: $p-b-2x=0$ $\Rightarrow px-x(b+x)$

(note: This is for the case $x>0$). If the derivative of profit is ≤ 0 at $x=0$ then $x=0$ is optimal. This is true when $p \leq b$.

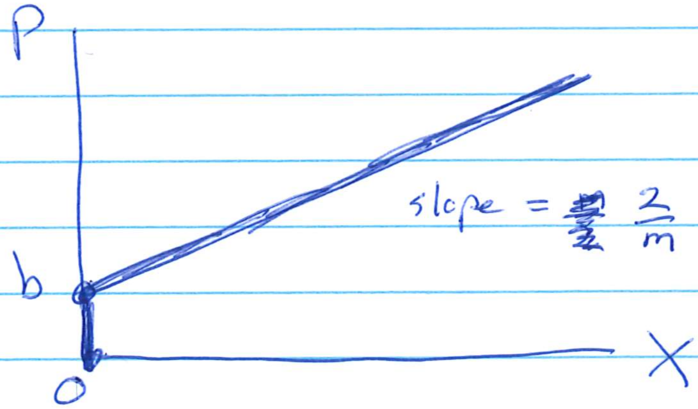
Note: SOC sufficient condition for profit max holds.

So the full supply function for a firm is:

$$x = 0 \text{ for } p \leq b \text{ and } x = \frac{p-b}{2} \text{ for } p \geq b$$

For market supply multiply by m firms to get

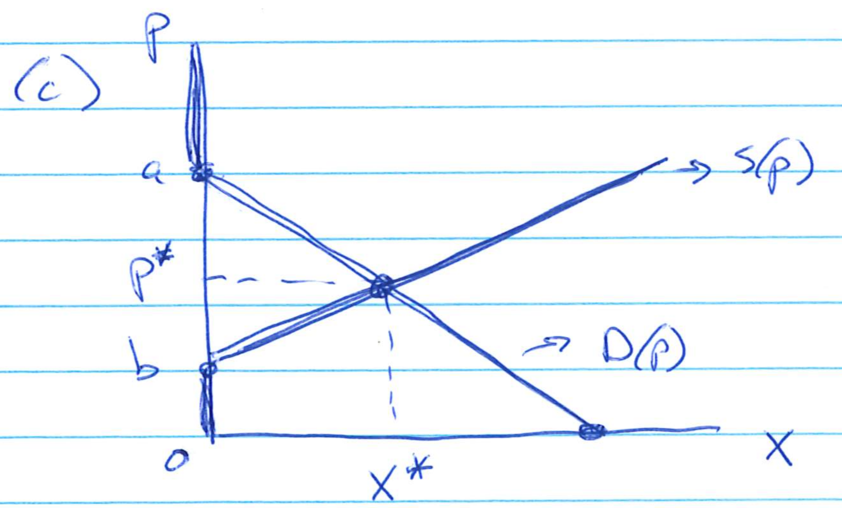
$$\begin{cases} X = 0 & \text{for } p \leq b \\ X = m\left(\frac{p-b}{2}\right) & \text{for } p \geq b \end{cases}$$



market supply curve for X

An individual firm cannot have negative profit. If $p \leq b$, the firm produces zero and has zero profit. If $p > b$, it has $P = MC > AC$ where $MC = b + 2x$ and $AC = b + x$. In this case, output is positive and profit is positive.

(c)



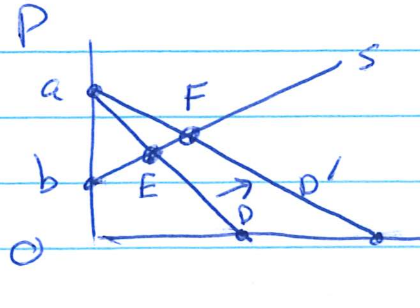
Set $D(p) = S(p)$ or

$$\frac{n(a-p)}{2} = \frac{m(p-b)}{2}$$

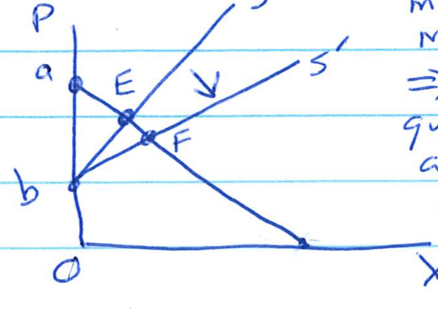
$$\Rightarrow na + mb = p(m+n)$$

$$\Rightarrow p^* = \frac{na+mb}{m+n}$$

where $X^* = D(p^*) = S(p^*)$



n increases ⇒ more demand ⇒ price rises, quantity rises as we go from E to F



m increases ⇒ more supply ⇒ price falls, quantity rises as we go from E to F

5. (a) Aggregate excess demand function:

$$z(p) = \sum_{i=1}^{n} x_i(p, pw_i) - \sum_{i=1}^{n} w_i$$

where $z(p) = [z_1(p) \ldots z_k(p)]$ is a k-dimensional vector;

$x_i(p, pw_i) = [x_{i1}(p, pw_i) \ldots x_{ik}(p, pw_i)]$ is consumer i's vector of Marshallian demands;

and $w_i = (w_{i1} \ldots w_{ik})$ is consumer i's endowment vector.

A Walrasian equilibrium (WE) is a price vector p^* such that $z(p^*) \leq 0$. Note that excess supply is allowed for one or more goods, but excess demand is not. The equilibrium allocation $x^* = (x_1^* \ldots x_n^*)$ is obtained from the Marshallian demands. Suppose p^* satisfies $z(p^*) \leq 0$. Then we also have $z(p^*) = z(tp^*) \leq 0$ for all $t > 0$. This is true because p has no effect on the endowment vectors, the Marshallian demands $x_i(p, m_i)$ are homogeneous of degree zero in prices and income, and the incomes are $m_i = pw_i$ for all i. Therefore tp^* is also a WE. Thus we cannot solve for the absolute price level (only relative prices matter).

(b) Walras's Law says $pz(p) \equiv 0$ for all p. Writing this out, it says $\sum_{j=1}^{k} p_j z_j(p) = 0$. To prove this, substitute the aggregate excess demand for each good j:

$$\sum_{j=1}^{k} p_j \left[\sum_{i=1}^{n} x_{ij}(p, pw_i) - \sum_{i=1}^{n} w_{ij}\right]$$

$$= \sum_{i=1}^{n} \underbrace{\left[\sum_{j=1}^{k} p_j x_{ij}(p, pw_i) - \sum_{j=1}^{k} p_j w_{ij}\right]}_{\text{This is zero for each } i \text{ because the budget constraints all hold with equality (marginal utilities are positive for all goods, so all income is spent)}} = 0$$

Suppose $p^* > 0$ and consider $\sum_{j=1}^{k} p_j^* z_j(p^*) = 0$.
In equilibrium, $z_j(p^*) \leq 0$ for all j so $p_j^* z_j^*(p) \leq 0$ for all j. If any j has $z_j(p^*) < 0$, then $p_j^* z_j(p^*) < 0$ for that j. This implies $\sum_{j=1}^{k} p_j^* z_j(p^*) < 0$ but this violates Walras's Law.
Therefore $p^* > 0$ implies $z_j(p^*) = 0$ for all j.

(c) The planner maxes $\sum_{i=1}^{n} a_i u_i(x_i)$ subject to the physical constraints $\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} w_{ij}$ for all j.

The Lagrangian is $L = \sum_{i=1}^{n} a_i u_i(x_i) - \sum_{j=1}^{k} q_j \left[\sum_{i=1}^{n} x_{ij} - \sum_{i=1}^{n} w_{ij}\right]$

FOC: $\frac{\partial L}{\partial x_{ij}} = a_i \frac{\partial u_i(x_i^*)}{\partial x_{ij}} - q_j = 0$ for all i, j
↳ and also the constraint equations.

Note: The objective function is strictly concave, so the FOC are sufficient and the solution x^* is unique.

In WE consumer i solves max $u_i(x_i)$ subject to $px_i = pw_i$
Lagrangian is $L_i = u_i(x_i) - d_i \left[\sum_j p_j x_{ij} - \sum_j p_j w_{ij}\right]$
FOC: $\frac{\partial L_i}{\partial x_{ij}} = \frac{\partial u_i(x_i)}{\partial x_{ij}} - d_i p_j = 0$ for all i, j.

We get the same allocation x^* when $p_j^* = q_j$ for all j and $d_i = \frac{1}{a_i}$ for all i.
[Note: all markets clear because x^* satisfies the planner's constraints]
We have to choose endowment vectors such that $p^* x_i^* = p^* w_i$ for all i, so each consumer can afford her equilibrium bundle (note that $w_i = x_i^*$ will work).
Yes, the WE is Pareto efficient because x^* solves the planner's problem. This implies that x^* is PE.
(Otherwise it could not be a solution – the planner could do better)